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In [1, 2] oscillations of a rod with current flowing along its surface were studied. A dispersion equation was obtained for longitudinal and transverse oscillations. We consider elastic oscillations of an infinite rod with current flowing along its surface when a uniform constant longitudinal magnetic field is present outside the rod. The dispersion equation for propagation of elastic oscillations is obtained and particular cases of longitudinal and transverse oscillations, and surface waves as well, are considered.

§1. Statement of the Problem and Boundary Conditions. Let a constant current I flow along the surface of an ideally conducting rod of radius a; outside the rod there is a uniform constant longitudinal magnetic field. Then field strength vector **H** in cylindrical coordinate system r, φ, and z has components

$$H_r = 0, \quad H_\varphi = \frac{2I}{cr}, \quad H_z = \text{const} \quad \text{for } r \geqslant a,$$
$$H_r = H_\varphi = H_z = 0 \quad \text{for } r < a.$$

The field **H** generates a magnetic pressure $p = 1/8\, H^2/\pi$ on the surface of the rod.

The vector $\mathbf{u}(u_r, u_\varphi, u_z)$ for the shift in rod points satisfies [3]

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\, \text{grad div}\, \mathbf{u} - \mu\, \text{rot rot}\, \mathbf{u}\,. \tag{1.1}$$

Here ρ is the density of the material, and λ, and μ are Lamè constants. The general solution to Eq. (1) has the form [1]

$$\mathbf{u} = \mathbf{U}(r)\, e^{i(-\omega t + m\varphi + kz)},$$
$$\mathbf{U}(r) = U(r)\,\mathbf{r}^\circ + V(r)\,\varphi^\circ + W(r)\,\mathbf{z}^\circ,$$
$$U(r) = A\frac{dJ_m(\alpha r)}{dr} + Bk\frac{dJ_m(\beta r)}{dr} + Cm\frac{J_m(\beta r)}{r}$$
$$\left(\alpha^2 = \frac{\rho\omega^2}{\lambda + 2\mu} - k^2\right), \tag{1.2}$$
$$V(r) = Aim\frac{J_m(\alpha r)}{r} + Bikm\frac{J_m(\beta r)}{r} + Ci\frac{dJ_m(\beta r)}{dr}$$
$$\left(\beta^2 = \frac{\rho\omega^2}{\mu} - k^2\right),$$
$$W(r) = AikJ_m(\alpha r) - Bi\beta^2 J_m(\beta r)\,. \tag{1.3}$$

Here A, B, and C are arbitrary constants.

Boundary conditions on the perturbed surface S' are as follows:

a) because the rod material is ideally conducting, the normal component of the field H' is zero

$$\mathbf{H}'\cdot\mathbf{n}' = 0; \tag{1.4}$$

b) the sums of magnetic pressure p outside the rod and stresses within the rod in projections on the r, φ, and z axies are zero,

$${}^1/_8 H^2 n_i' / \pi + \sigma_{ik}' n_k' = 0 \quad (i, k = r, \varphi, z)\,. \tag{1.5}$$

Here **n'** is the external normal to S'; n_i' are its components; $\mathbf{H}' = \mathbf{H} + \mathbf{h}$ is the excited magnetic field; $\sigma_{ik}' = \sigma_{ik}^\circ + \sigma_{ik}$ is the stress where σ_{ik}° is the stress due to the effect of the field **H**;

$$\overset{\circ}{\sigma}_{rr} = \overset{\circ}{\sigma}_{\varphi\varphi} = -{}^1/_8 H^2/\pi, \qquad \overset{\circ}{\sigma}_{zz} = \overset{\circ}{\sigma}_{r\varphi} = \overset{\circ}{\sigma}_{\varphi z} = \overset{\circ}{\sigma}_{zr} = 0. \tag{1.6}$$

The external normal **n'** to S' is written approximately as

$$\mathbf{n}' = \mathbf{n}^\circ - \nabla u_r, \quad \text{or} \quad \mathbf{n}' = \mathbf{r}^\circ - im\frac{u_r}{a}\varphi^\circ - iku_r\mathbf{z}^\circ. \tag{1.7}$$

Here $\mathbf{r}^\circ$, φ°, and $\mathbf{z}^\circ$ are unit vectors; on surfaces S° (for r = a), u_r is a function only of φ and z.

On the surface S' we have $r = a + u_r(a, \varphi, z)$; therefore, for field H' or S', we obtain

$$\mathbf{H}' = \mathbf{H}'(a) + [(u_r\mathbf{r}^\circ\cdot\nabla)\,\mathbf{H}']_{r=a} \quad \text{or}$$

$$\mathbf{H}' = \mathbf{H}(a) + \mathbf{h}(a, \varphi, z) - u_r(a, \varphi, z)\, a^{-1} H_\varphi(a)\, \varphi^\circ, \tag{1.8}$$

which is accurate to within terms of the first order of smallness with respect to $|\mathbf{u}|$.

It follows from the Maxwell equations that div **h** = 0 and rot **h** = = 0 outside the rod. We now have

$$\mathbf{h} = -iL\nabla K_m(kr)\, e^{i(-\omega t + m\varphi + kz)}. \tag{1.9}$$

Here $K_m(kr)$ is a modified m-th order Bessel function of the second kind. The arbitrary constant L is such that the field vanishes at infinity. Within the rod the field is zero. From (1.8) and (1.9), with the same accuracy, we have

$$(H')^2 = H_\varphi^2 + H_z^2 + 2La^{-1}(mH_\varphi + kaH_z)\, K_m(ka)\, e^{i(-\omega t + m\varphi + kz)} - 2u_r a^{-1} H_\varphi^2 \tag{1.10}$$

Substituting (1.6), (1.7), (1.8), (1.9) and (1.10) into (1.4) and (1.5) we obtain

$$(mH_\varphi + kaH_z)\, a^{-1}u_r + LkK_m'(ka)\, e^{i(-\omega t + m\varphi + kz)} = 0,$$
$$\frac{H_\varphi^2 u_r}{4\pi a} - \frac{mH_\varphi + kaH_z}{4\pi a} LK_m(ka)\, e^{i(-\omega t + m\varphi + kz)} - \sigma_{rr} = 0, \tag{1.11}$$
$$\sigma_{r\varphi} = 0, \qquad \frac{H_\varphi^2 + H_z^2}{8\pi} iku_r - \sigma_{rz} = 0.$$

All quantities in (1.10) and (1.11) are taken for r = a.

Using (1.3) for amplitudes of the displacement vector and relationships between stresses and elastic deformations, we obtain the following system from (1.11) for finding the constants:

$$b_{i1}\frac{A}{a^2} + b_{i2}\frac{B}{a^3} + b_{i3}\frac{C}{a^2} + b_{i4}\frac{L}{a\sqrt{8\pi E}} = 0 \tag{1.12}$$
$$(i = 1, 2, 3, 4)\,.$$

Here E is Yong's modulus and the coefficients b_{ij} are found from

$$\begin{aligned}
&b_{11} = \alpha a\gamma J_m'(\alpha a), \quad b_{12} = a^2\beta\gamma kJ_m'(\beta a),\\
&b_{13} = \gamma mJ_m(\beta a), \quad b_{14} = kaK_m'(ka),\\
&b_{21} = \alpha a\delta EJ_m'(\alpha a) + [a^2\rho\omega^2 - 2\mu\,(k^2a^2 + m^2)]J_m(\alpha a),\\
&b_{22} = a^2\beta\delta EkJ_m'(\beta a) + 2\mu ka\,(a^2\beta^2 - m^2)J_m(\beta a),\\
&b_{23} = \delta EmJ_m(\beta a) - 2\mu\beta amJ_m'(\beta a),\\
&b_{24} = -2\gamma\, EK_m(ka), \quad b_{31} = m\,[\alpha aJ_m'(\alpha a) - J_m(\alpha a)],\\
&b_{32} = kam\,[\beta aJ_m'(\beta a)] - J_m(\beta a),\\
&b_{33} = (m^2 - {}^1/_2 a^2\beta^2)J_m(\beta a) - \beta aJ_m'(\beta a),\\
&b_{34} = 0, \quad b_{41} = (\theta\, E - 2\mu)\,\alpha\, ka^2J_m'(\alpha a),\\
&b_{42} = [(\theta\, E - \mu)\, k^2 + \mu\beta^2]\,\beta a^3J_m'(\beta a),\\
&b_{43} = (\theta\, E - \mu)\, kamJ_m(\beta a), \quad b_{44} = 0,\\
&\gamma = \frac{mH_\varphi + kaH_z}{\sqrt{8\pi E}}, \qquad \delta = \frac{1}{E}\left(\frac{H_\varphi^2}{4\pi} + 2\mu\right),\\
&\theta = \frac{H_\varphi^2 + H_z^2}{8\pi E}.
\end{aligned} \tag{1.13}$$

In the Bessel functions, the primes denote derivatives with respect to the arguments αr and βr for r = a.

§2. Dispersion Equation. System (1.12) has a nontrivial solution because its determinant must be zero:

$$|b_{ij}| = 0 \qquad (i, j = 1, 2, 3, 4)\,. \tag{2.1}$$

We introduce the dimensionless quantities:

$$x = ka, \quad y^2 = \frac{\rho\omega^2}{k^2E}, \quad h_\varphi^2 = \frac{H_\varphi^2}{8\pi E}, \quad h_z^2 = \frac{H_z^2}{8\pi E}$$
$$X = \alpha a = x\left[\frac{(1+\nu)(1-2\nu)}{1-\nu}y^2 - 1\right]^{1/2}. \tag{2.2}$$
$$Y = \beta a = [2(1+\nu)\,y^2 - 1]^{1/2}\, x$$

Here γ is the Poisson coefficient. The elements of the determinant take the form

$$b_{11} = \gamma XJ_m'(X), \quad b_{12} = \gamma xYJ_m'(Y),$$

$$
\begin{aligned}
&b_{13} = \gamma m J_m'(Y),\ b_{14} = xK_m'(x),\\
&b_{21} = \{\delta X J_m'(X) + (1+\nu)^{-1}[{}^1/_2(Y^2 - x^2) - m^2]J_m(X)\}E,\\
&b_{22} = \{\delta x Y J_m'(Y) + x(1+\nu)^{-1}(Y^2 - m^2)J_m(Y)\}E,\\
&b_{23} = \{\delta m J_m(Y) - m(1+\nu)^{-1}YJ_m(Y)\}E,\\
&b_{24} = -2\gamma E K_m(x),\ b_{31} = m[XJ_m'(X) - J_m(X)],\\
&b_{32} = mx[YJ_m'(Y) - J_m(Y)],\\
&b_{33} = (m^2 - {}^1/_2 Y^2)J_m(Y) - YJ_m'(Y),\\
&b_{34} = 0,\ b_{41} = [\theta - (1+\nu)^{-1}]\,xEXJ_m'(X),\\
&b_{42} = \left(\theta - \frac{Y^2 - x^2}{2x^2(1+\nu)}\right)x^2EYJ_m'(Y),\\
&b_{43} = \left(\theta - \frac{1}{2(1+\nu)}\right)xmEJ_m(Y),\\
&b_{44} = 0,\ \gamma = mh_\varphi + xh_z,\\
&\delta = 2h_\varphi^2 + (1+\nu)^{-1},\ \theta = h_\varphi^2 + h_z^2.
\end{aligned}
\tag{2.3}
$$

From the first row of determinant (2.1) with elements (2.3) we eliminate γ while from the first, second, third, and fourth columns, we eliminate, respectively,

$$
XJ_m'(X),\quad YJ_m'(Y),\quad mJ_m(Y),\quad \gamma^{-1}xK_m'(x).
$$

After this is done we introduce notation

$$
\varphi_m(\xi) = \frac{J_m(\xi)}{\xi J_m'(\xi)},\qquad \psi_m(\xi) = \frac{K_m(\xi)}{\xi K_m'(\xi)}, \tag{2.4}
$$

which simplifies transformations and expansions in the elements of the first row, the left side of dispersion equation (2.1) is written as the third-order determinant

$$
|c_{ij}| = 0 \quad (i, j = 1, 2, 3) \tag{2.5}
$$

$$
\begin{aligned}
&c_{11} = 1 + 2(1+\nu)[h^2_\varphi + \gamma^2\psi_m(x)] +\\
&+ [{}^1/_2(Y^2 - x^2) - m^2]\varphi_m(X),\\
&c_{12} = (Y^2 - m^2)\varphi_m(Y) - [{}^1/_2(Y^2 - x^2) - m^2]\varphi_m(X),\\
&c_{13} = -\varphi_m^{-1}(Y) - [{}^1/_2(Y^2 - x^2) - m^2]\varphi_m(X),\\
&c_{21} = m^2[1 - \varphi_m(X)],\ c_{22} = m^2[\varphi_m(X) - \varphi_m(Y)],\\
&c_{23} = m^2\varphi_m(X) - \varphi_m^{-1}(Y) - {}^1/_2 Y^2,\\
&c_{31} = 2(1+\nu)\theta - 2,\\
&c_{32} = 1 - Y^2/x^2,\ c_{33} = 1.
\end{aligned}
\tag{2.6}
$$

§3. Longitudinal Oscillations (Constrictions). These oscillations will occur if we set m = 0 in (2.5) and (2.6). The left side of (2.5) is expanded in powers of elements in the second column and, by making some transformations, we obtain

$$
|d_{ij}| = 0 \qquad (i, j = 1, 2), \tag{3.1}
$$

$$
\begin{aligned}
&d_{11} = 1 + 2(1+\nu)[h^2_\varphi + x^2\psi_0(x)\,h^2_z] + Y^2\varphi_0(Y),\\
&d_{12} = Y^2\varphi_0(Y) - {}^1/_2(Y^2 - x^2)\varphi_0(X)\quad d_{22} = 1 + Y^2/x^2,\\
&d_{21} = 2(1+\nu)(h_\varphi^2 + h_z^2) + Y^2/x^2 - 1.
\end{aligned}
\tag{3.2}
$$

Assume the long-wave situation exists ($x \ll 1$]; then (2.4) can be written (for $\xi \ll 1$) as

$$
\varphi_0 \approx {}^1/_4 - 2/\xi^2,\ \ \psi_0(\xi) \approx C + \ln {}^1/_2\,\xi, \tag{3.3}
$$

where C is the Euler constant.

If we substitute (3.3) into (3.2) and (3.1), ignore terms greater terms greater than the second order of smallness and return to the variables y^2 and ν, the equation for long-wave longitudinal oscillations is

$$
y^2 = \frac{1 - 2h_\varphi^2 - 2[\nu + (1-\nu)x^2\ln({}^1/_2x)]h_z^2}{1 - 2(1+\nu)(1-2\nu)[h_\varphi^2 + x^2h_z^2\ln({}^1/_2x)]}. \tag{3.4}
$$

§4. Transverse Oscillations. a) For m = ±1 we have transverse oscillations with the same cross section. Equation (2.5) with elements (2.6) is written as

$$
|e_{ij}| = 0 \qquad (i, j = 1, 2, 3), \tag{4.1}
$$

$$
\begin{aligned}
&e_{11} = 1 + 2(1+\nu)\ [h_\varphi^2 + (\pm h_\varphi + xh_z)^2\psi_1(x)] - \varphi_1^{-1}(Y),\\
&e_{12} = \varphi_1^{-1}(Y) + (Y^2 - 1)\varphi_1(Y),\\
&e_{13} = [1 + {}^1/_2(x^2 - Y^2)]\varphi_1(X) - \varphi_1^{-1}(Y),\\
&e_{21} = 1 - {}^1/_2 Y^2 - \varphi_1^{-1}(Y),\\
&e_{22} = {}^1/_2 Y^2 - \varphi_1(Y) + \varphi_1^{-1}(Y),\\
&e_{23} = \varphi_1(X) - \varphi_1^{-1}(Y) - {}^1/_2 Y^2,\\
&e_{31} = 2(1+\nu)(h_\varphi^2 + h_z^2) - 1,\\
&e_{32} = Y^2/x^2,\quad e_{33} = 1,\\
&\varphi_1(\xi) = \varphi_{-1}(\xi),\quad \psi_1(\xi) = \psi_{-1}(\xi).
\end{aligned}
$$

Let the wavelength be greater than the rod diameter ($x \ll 1$); then

$$
\begin{aligned}
&\varphi_1(\xi) \approx 1 + {}^1/_4\xi^2,\ \varphi_1^{-1}(\xi) \approx 1 - {}^1/_4\xi^2,\\
&\psi_1(\xi) \approx -1 - \xi^2(C + \ln {}^1/_2\xi).
\end{aligned}
\tag{4.3}
$$

By substituting (4.3) into (4.2) and making some transformations involving (4.1), we obtain

$$
|g_{ij}| = 0 \qquad (i, j = 1, 2, 3),
$$

$$
\begin{aligned}
&g_{11} = Y^2 + 4(1+\nu)x^{-2}[h_\varphi^2 + (\pm h_\varphi + xh_z)^2\psi_1(x)],\\
&g_{12} = 1 + {}^1/_2Y^2,\ \ g_{13} = {}^1/_4(X^2 - 2Y^2 - x^{-2}X^2Y^2),\\
&g_{21} = -Y^2,\quad g_{22} = 0,\quad g_{23} = X^2 - Y^2,\\
&g_{31} = 2(1+\nu)(h^2_\varphi + h_z^2) - 1,\ g_{32} = 1,\ g_{33} = 0.
\end{aligned}
$$

We expand (4.4), substitute g_{ij} and X and Y in (2.2), and ignore terms greater than the second order of smallness. The equation is then found in terms of

$$
\begin{aligned}
y^2 = {}^1/_4x^2 + \{1 - 2x^{-2}[1 + \psi_1(x)]\}h_\varphi^2 &+\\
+ [1 - 2\psi_1(x)]h_z^2 \mp 4x^{-1}h_\varphi h_z\psi_1(x)&
\end{aligned}
\tag{4.6}
$$

or

$$
\begin{aligned}
\frac{\rho\omega^2}{k^2E} &= {}^1/_4k^2a^2 + 2[{}^1/_2 + C + \ln({}^1/_2ka)]h_\varphi^2 +\\
&+ 2k^2a^2[3k^{-2}a^{-2} + C + \ln({}^1/_2ka)]h_z^2 \mp\\
&\mp 4ka[k^{-2}a^{-2} + C + \ln({}^1/_2ka)]h_\varphi h_z.
\end{aligned}
\tag{4.7}
$$

b) For m = ±2, we also have transverse oscillations in the rod, but with a different cross sectional shape, and (2.5) now becomes

$$
|p_{ij}| = 0 \qquad (i, j = 1, 2, 3),
$$

$$
\begin{aligned}
&p_{11} = 1 + 2(1+\nu)[h_\varphi^2 +\\
&+ (\pm 2h_\varphi + xh_z)^2\psi_2(x)] - \varphi_2^{-1}(Y),\\
&p_{12} = (Y^2 - 4)\varphi_2(Y) + \varphi_2^{-1}(Y),\\
&p_{13} = -\varphi_2^{-1}(Y) - {}^1/_2(Y^2 - x^2 - 8)\varphi_2(X),\\
&p_{21} = 4 - {}^1/_2Y^2 - \varphi_2^{-1}(Y),\\
&p_{22} = {}^1/_2Y^2 + \varphi_2^{-1}(Y) - 4\varphi_2(Y),\\
&p_{23} = 4\varphi_2(X) - {}^1/_2Y^2 - \varphi_2^{-1}(Y)\\
&p_{31} = 2(1+\nu)(h_\varphi^2 + h_z^2) - 1,\ p_{32} = Y^2/x^2,\\
&p_{33} = 1,\ \varphi_2(\xi) = \varphi_{-2}(\xi),\ \psi_2(\xi) = \psi_{-2}(\xi).
\end{aligned}
$$

For small ξ, functions $\varphi_2(\xi)$ and $\psi_2(\xi)$ are, approximately,

$$
\begin{aligned}
&\varphi_2(\xi) \approx {}^1/_2 + {}^1/_{24}\xi^2,\ \varphi_2^{-1}(\xi) \approx 2 - {}^1/_6\xi^2,\\
&\psi_2(\xi) \approx -{}^1/_2 + {}^1/_8\xi^2.
\end{aligned}
\tag{4.10}
$$

Using (4.10), we substitute $\varphi_2(\xi)$ and $\psi_2(\xi)$ into (4.9). We then obtain

$$
|q_{ij}| = 0 \qquad (i, j = 1, 2, 3), \tag{4.11}
$$

$$
\begin{aligned}
&q_{11} = {}^1/_2Y^2 - 3 + 2(1+\nu)[h_\varphi^2 + (\pm 2h_\varphi + xh_z)^2\psi_2(x)],\\
&q_{12} = {}^1/_{24}Y^2,\ q_{13} = {}^1/_4(Y^2 + x^2) - {}^1/_{48}X^2(Y^2 - x^2),\\
&q_{21} = 12 - 2Y^2,\ q_{22} = 1,\ q_{23} = X^2 - 2Y^2,\\
&q_{31} = 2(1+\nu)(h_\varphi^2 + h_z^2) - 1,\ q_{32} = 1/x^2,\ q_{33} = 1
\end{aligned}
\tag{4.12}
$$

We expand the determinant on the left side of (4.11) with elements (4.12) and ignore terms greater than the second order of smallness; upon solving we obtain

$$
\begin{aligned}
y^2 = \frac{2}{1+2\nu}\Big\{&\frac{\nu + 3x^{-2}}{1+\nu} + [\nu - 2 + 2(3 - 2\nu)x^{-2}]h_\varphi^2 +\\
&+ [(4 - 3\nu) - {}^1/_4(3 - 2\nu)x^2]h_z^2 \mp (3 - 2\nu)(x - 4x^{-1})h_\varphi h_z\Big\}
\end{aligned}
\tag{4.13}
$$

or

$$
\begin{aligned}
\frac{\rho\omega^2}{k^2E} = \frac{2}{1+2\nu}\Big\{&\frac{\nu + 3k^{-2}a^{-2}}{1+\nu} + [\nu - 2 + 2(3 - 2\nu)k^{-2}a^{-2}]h_\varphi^2 +\\
&+ [4 - 3\nu - {}^1/_4(3 - 2\nu)k^2a^2]h_z^2 \mp\\
&\pm (3 - 2\nu)(ka - 4k^{-1}a^{-1})h_\varphi h_z\Big\}.
\end{aligned}
\tag{4.14}
$$

§5. **Surface Waves.** For surface waves $x \gg 1$, the quantities X and Y will be imaginary; then [4]

$$\varphi_m(i\xi) \approx 1/\xi,\ \psi_m(\xi) \approx -1/\xi \quad (\xi \gg 1). \tag{5.1}$$

After substituting in (5.1) and ignoring terms on the order of $1/\xi$, Eq. (2.5) with elements (2.6) becomes

$$\begin{aligned} &\{1-(1+\nu)y^2-[(1-2(1+\nu)y^2)\times \\ &\times(1-(1+\nu)(1-2\nu)(1-\nu)^{-1}y^2)]^{1/2}\}h_\varphi^2+ \\ &+\{1-(1+\nu)y^2+[2(1+\nu)y^2-(1-2(1+\nu)y^2)^{1/2}]\times \\ &\times[1-(1+\nu)(1-2\nu)(1-\nu)^{-1}y^2]^{1/2}\}h_z^2+ \\ &+(1+\nu)^{-1}\{[(-1-2(1+\nu)y^2)\times \\ &\times(1-(1+\nu)(1-2\nu)(1-\nu)^{-1}y^2)]^{1/2}- \\ &-[1-(1+\nu)y^2]^2\}=0. \end{aligned} \tag{5.2}$$

Letting $h_\varphi = h_z = 0$ in (5.2) we obtain a relationship for Rayleigh surface waves [3].

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